Accuracy of Range-Based Localization Schemes in Random Sensor Networks: A Lower Bound Analysis

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***Abstract*— Accuracy is a fundamental performance requirement in network localization. This paper studies the accuracy of range-based localization schemes for random sensor networks with respect to network connectivity and scale. We show that the variance of localization errors is proportional to the average geometric dilution of precision (AGDOP). The paper proves a novel lower bound of expectation of AGDOP (LB-E-AGDOP). Our analysis based on LB-E-AGDOP shows that localization accuracy is approximately inversely proportional to the average degree of network. A further analysis shows that when network connectivity merely guarantees localizability, increasing sensor nodes leads to bounded monotonic increase in AGDOP; when a network is densely connected, increasing sensor nodes leads to bounded monotonic decrease in AGDOP. Finally, these conclusions are validated by numerical simulations.**

I. Introduction

Robotic sensor networks, namely sensor networks with robotic mobility, represent a new paradigm of large-scale, flexible, robust, cost-effective data collection and information processing in complex environments. They are expected to enable many fascinating applications including assisted navigation and surveillance, wildlife habitat monitoring, oceanographic data collection, and disaster management [1]–[4]. Many of these applications rely on accurate location information about sensor nodes. To this end, various network localization schemes have been explored over the past decade. These schemes can be generally classified into five categories: range-based [5], [6], angle-based [7], [8], proximity-based [9], [10], event-driven [11], [12], and simultaneous localization and map building [13], [14]. In this paper, we focus on range-based schemes because they can achieve better localization accuracy than most other schemes [15].

Fig. 1 illustrates a scenario of range-based network localization. Since range-based network localization is essentially a graph realization problem [16]–[18], connectivity of the graph exerts significant influence on many performance metrics, such as accuracy, energy efficiency, localizability, robustness, and scalability. Although localizability has been studied with respect to connectivity [16], [17], the relationship between accuracy and connectivity has not yet been theoretically treated. This paper aims at a generalized theory to characterize accuracy with respect to connectivity. Specifically, the following problems are addressed in this paper.

- What is the quantitative relationship between localization accuracy and network connectivity?
- For a certain level of connectivity, how does accuracy vary with network scale (number of sensor nodes)?

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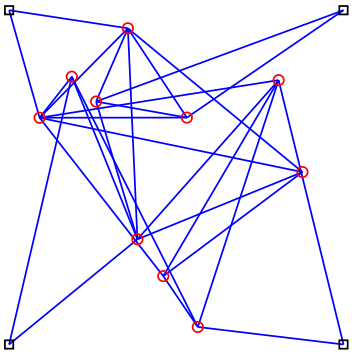

Fig. 1. A scenario of range-based network localization in 2 dimensions. Squares denote anchor nodes whose locations are known, circles denote sensor nodes whose locations are to be estimated, and lines represent ranging links which provide inter-node distance information. We assume that sensor nodes are randomly distributed, and ranging links are randomly established to achieve a certain level of connectivity. The purpose of this paper is to study the relationship between localization accuracy and network connectivity.

In this paper, connectivity is defined as average sensor and node degrees of the graph. Connectivity affects geometry of nodes. Geometry and inter-node ranging accuracy together determine localization accuracy. We first show that the variance of localization errors is proportional to the average geometric dilution of precision (AGDOP) and thus decouple localization accuracy from range accuracy. Next, we prove a lower bound of E-AGDOP (LB-E-AGDOP), and obtain a closed-form expression (19) that characterizes overall network localization accuracy with respect to connectivity and network scale. The closed-form expression is then used to answer the above two questions. The theoretical answers are validated by numerical simulations.

The rest of this paper is organized as follows. Section II formulates the network localization problem and introduces the connectivity metrics used throughout this paper. Section III analyzes localization accuracy and its relationship to AGDOP. Section IV derives a closed-form expression for LB-E-AGDOP with respect to network connectivity and scale. Section V studies how location accuracy varies with the network scale. Section VI presents numerical simulation results to validate the theoretical conclusions. Finally, Section VII concludes the paper.

II. Preliminaries

A. Problem formulation

In this paper, a sensor network is modeled as a *simple* graph[1] $G = (V, E)$, where $V = \{1, 2, \ldots, N\}$ is a set of N

[1]A simple graph, also referred to as a strict graph, is an unweighted, undirected graph containing no self-loops or multiple edges [19], [20].

nodes (or vertices), and $E = \{e_1, e_2, \ldots, e_K\} \subseteq V \times V$ is a set of K links (or edges) that connect the nodes [17].

All nodes are in a d-dimensional Euclidean space ($d \geq 1$), with the locations denoted by $p_n \in \mathbb{R}^d$, $n = 1, \ldots, N$. The first N_S nodes, labeled 1 through N_S, are ordinary *sensor* nodes (or mobile nodes), whose locations are unknown; the rest $N_A = N - N_S$ nodes, labeled $N_S + 1$ through N, are special *anchor* nodes (or beacon nodes), who are aware of their locations, either through GPS or manual pre-programming during deployment.

An unordered pair $e_k = (i_k, j_k) \in E$ if and only if there exists a direct ranging link between nodes i_k and j_k. The link provides inter-node distance information $\rho_k = r_k + \epsilon_k$, where $r_k = \|p_{i_k} - p_{j_k}\|$ is the actual distance between nodes i_k and j_k, and ϵ_k is the range measurement error. The range measurements ρ_k can be obtained by a variety of methods, such as one-way time of arrival (ToA), two-way ToA, and received signal strength indication [21]. In this paper, we assume perfect clock synchronization, i.e., zero clock biases, if the range is measured by one-way ToA.

The network localization problem is to determine the locations of sensor nodes p_n, $n = 1, \ldots, N_S$, given a fixed network graph G, known locations of anchors p_n, $n = N_S+1, \ldots, N$, and range measurements ρ_k, $k = 1, \ldots, K$.

B. Metrics of connectivity

Assume no anchor-to-anchor links. For all nodes $n = 1, \ldots, N$, we define the following *degrees*:

- Anchor degree: $\deg_A(n)$, the number of anchor-to-sensor links incident to node n;
- Sensor degree: $\deg_S(n)$, the number of sensor-to-sensor links incident to node n;
- Degree: $\deg(n) = \deg_A(n) + \deg_S(n)$, the number of links incident to node n;

In graph theory, connectivity is usually described by vertex connectivity or edge connectivity. For example, a graph is κ-vertex/edge-connected if it remains connected whenever fewer than κ vertices/edges are removed [20]. Unfortunately, vertex/edge connectivity only describes some "minimum" properties of connectivity, such as $\min_n \deg(n)$ [20], and does not distinguish between sensor and anchor nodes. This paper uses average degrees to characterize the "average" connectivity of the network. The average degrees are defined as

$$\delta_* = \frac{1}{N_S} \sum_{n=1}^{N_S} \deg_*(n), \tag{1}$$

where the subscript $_*$ can be blank, $_A$, or $_S$, for the average degree, average anchor degree, or average sensor degree, respectively.

Let K_S and K_A denote the number of sensor-to-sensor and anchor-to-sensor links in the network, respectively. It is easy to verify the equalities $K = K_S + K_A$, $N_S\delta_S = 2K_S$, $N_S\delta_A = K_A$, and $\delta = \delta_S + \delta_A$.

III. Localization Accuracy

Localization is fundamentally an optimization problem that finds coordinate vectors $p_n \in \mathbb{R}^d$, $n = 1, \ldots, N_S$, such that for each ranging link $e_k = (i_k, j_k) \in E$, the distance $r_k = \|p_{i_k} - p_{j_k}\|$ is as close to the range measurement ρ_k as possible.

Assume that range errors follow a zero-mean Gaussian distribution, i.e.,

$$\rho_k = r_k + \epsilon_k, \quad \epsilon_k \sim \mathcal{N}(0, \sigma_k^2), \quad \forall k = 1, \ldots, K. \tag{2}$$

Then, the maximum likelihood (ML) estimation of $\{p_n\}_{n=1}^{N_S}$ is equivalent to the weighted least squares (LS) problem

$$\min_{\{p_n\}_{n=1}^{N_S}} \sum_{k=1}^{K} \frac{(\|p_{i_k} - p_{j_k}\| - \rho_k)^2}{\sigma_k^2}. \tag{3}$$

The LS problem cannot be directly solved because the distance $r_k = \|p_{i_k} - p_{j_k}\|$ is a nonlinear function of the coordinate vectors p_{i_k} and p_{j_k}. Let $\boldsymbol{p} = \mathrm{column}\{p_1, p_2, \ldots, p_{N_S}\} \in \mathbb{R}^{dN_S}$ and $\boldsymbol{r}(\boldsymbol{p}) = (r_1(\boldsymbol{p}), r_2(\boldsymbol{p}), \ldots, r_K(\boldsymbol{p}))^{\mathrm{T}} \in \mathbb{R}^K$. The first-order linear approximation of the distance function $\boldsymbol{r}(\boldsymbol{p})$ with respect to an initial guess $\boldsymbol{p}_0$ can be written as

$$\boldsymbol{r}(\boldsymbol{p}_0 + \Delta\boldsymbol{p}) = \boldsymbol{r}(\boldsymbol{p}_0) + G\Delta\boldsymbol{p}, \tag{4}$$

where the *geometry matrix* $G \in \mathbb{R}^{K \times dN_S}$ is given by

$$G = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{p}} = \begin{bmatrix} \frac{\partial r_1}{\partial p_{1,1}} & \cdots & \frac{\partial r_1}{\partial p_{1,d}} & \cdots & \frac{\partial r_1}{\partial p_{N_S,d}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial r_K}{\partial p_{1,1}} & \cdots & \frac{\partial r_K}{\partial p_{1,d}} & \cdots & \frac{\partial r_K}{\partial p_{N_S,d}} \end{bmatrix}, \tag{5}$$

where $p_{i,m}$, $m = 1, \ldots, d$, is the mth element of the coordinate vector p_i. Each element of the geometry matrix G is given by

$$G_{k,(n-1)d+m} = \frac{\partial r_k}{\partial p_{n,m}} = \begin{cases} \frac{p_{i_k,m} - p_{j_k,m}}{\|p_{i_k} - p_{j_k}\|} & \text{if } n = i_k, \\ \frac{p_{j_k,m} - p_{i_k,m}}{\|p_{i_k} - p_{j_k}\|} & \text{if } n = j_k, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Each row of G represents a link. There are only d nonzero elements in a row for an anchor-to-sensor link, and there are $2d$ nonzero elements for a sensor-to-sensor link. Therefore, G is highly sparse when the network contains many nodes.

Assume that the network is localizable. Then, G must be a tall matrix (i.e., $K \geq dN_S$ [16], [17], [22], [23]) with full column rank, and the weighted LS problem (3) can be solved by the following iterative algorithm, which is based on the Newton–Raphson method [24],

$$\boldsymbol{p}^{(n+1)} = \boldsymbol{p}^{(n)} + (G^{\mathrm{T}}\Sigma^{-1}G)^{-1}G^{\mathrm{T}}\Sigma^{-1}[\boldsymbol{\rho} - \boldsymbol{r}(\boldsymbol{p}^{(n)})], \tag{7}$$

where $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_K)^{\mathrm{T}}$, $\Sigma = \mathrm{cov}(\boldsymbol{\epsilon}, \boldsymbol{\epsilon})$ is the covariance of range errors, where $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_K)^{\mathrm{T}}$.

When the initial guess $\boldsymbol{p}^{(0)}$ is accurate enough and the iteration converges, the localization errors $\boldsymbol{\omega}$ have the following relationship to the range errors $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_K)^{\mathrm{T}}$:

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{p}^{(\infty)} - \boldsymbol{p} = (G^{\mathrm{T}}\Sigma^{-1}G)^{-1}G^{\mathrm{T}}\Sigma^{-1}[\boldsymbol{\rho} - \boldsymbol{r}(\boldsymbol{p})] \\ &= (G^{\mathrm{T}}\Sigma^{-1}G)^{-1}G^{\mathrm{T}}\Sigma^{-1}\boldsymbol{\epsilon}. \end{aligned} \tag{8}$$

The covariance of localization errors is thus given by

$$\begin{aligned} \mathrm{cov}(\boldsymbol{\omega}, \boldsymbol{\omega}) &= (G^{\mathrm{T}}\Sigma^{-1}G)^{-1}G^{\mathrm{T}}\Sigma^{-1}\,\mathrm{cov}(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}) \\ &\qquad \Sigma^{-1}G^{\mathrm{T}}(G^{\mathrm{T}}\Sigma^{-1}G)^{-1} \\ &= (G^{\mathrm{T}}\Sigma^{-1}G)^{-1}. \end{aligned} \tag{9}$$

This has achieved the Cramér-Rao bound [25], [26].

If range measurement errors are independent and identically distributed (iid), i.e., $\Sigma = \mathrm{diag}(\sigma^2, \ldots, \sigma^2)$, we have

$$\mathrm{cov}(\boldsymbol{\omega}, \boldsymbol{\omega}) = (G^{\mathrm{T}}\Sigma^{-1}G)^{-1} = \sigma^2(G^{\mathrm{T}}G)^{-1}. \tag{10}$$

The matrix $H = (G^{\mathrm{T}}G)^{-1} \in \mathbb{R}^{dN_S \times dN_S}$ is referred to as dilution of precision (DOP) matrix. DOP is a term widely used in satellite navigation specifying the multiplicative effect on positioning accuracy due to satellite geometry[2] [24]. For network localization, DOP specifies the multiplicative effect due to not only node geometry but also network connectivity. DOP decouples localization accuracy from range accuracy. The smaller DOP is, the better localization accuracy one can expect.

A diagonal element $H_{(n-1)d+m,(n-1)d+m}$ is the DOP of coordinate m for node n. The sum of all the diagonal elements, $\mathrm{trace}(H)$, is the geometric DOP (GDOP) of the whole network. In this paper, we define average GDOP (AGDOP) as GDOP divided by the number of sensor nodes, $\mathrm{trace}(H)/N_S$. AGDOP is a performance indicator of localization accuracy of each node due to network geometry and connectivity.

For a network with random node locations and random links, AGDOP is a random variable. The expectation of AGDOP (E-AGDOP) indicates the expected localization accuracy because the root-mean-square localization error is proportional to E-AGDOP. We shall use E-AGDOP and its lower bound to study the relationship between localization accuracy and network connectivity in the rest of the paper.

IV. Lower Bound of E-AGDOP

The value of E-AGDOP can be simply derived from the diagonal elements of $\mathrm{E}\,H = \mathrm{E}[(G^{\mathrm{T}}G)^{-1}]$. Unfortunately, it is very difficult to obtain a closed-form expression of $\mathrm{E}\,H$ for a randomly-deployed network (random node locations and random links) that achieves a certain level of connectivity. Instead of evaluating $\mathrm{E}\,H$, we consider $F = G^{\mathrm{T}}G$ in this paper, not only because $\mathrm{E}\,F$ can be evaluated analytically, but also because $(\mathrm{E}\,F)^{-1} = [\mathrm{E}(G^{\mathrm{T}}G)]^{-1}$ is proven to be a lower bound of $\mathrm{E}[(G^{\mathrm{T}}G)^{-1}]$, as detailed in Appendix I.

The matrix F is a function of node locations and links. In this paper, we assume that 1) the the location of sensor nodes in each dimension is iid, and 2) links are uniformly distributed and independent of node locations. Although the latter assumption is a bit far from many realistic scenarios, it is necessary for a simple theoretical development. Under the two assumptions, $\mathrm{E}\,F$ can be evaluated by the following two steps:

Step 1: $\Xi = \mathrm{E}_{\text{nodes}}(F|\text{links})$, conditional expectation of F for randomly-deployed nodes given certain links;

Step 2: $\mathrm{E}\,F = \mathrm{E}_{\text{links}}(\Xi)$, expectation of Ξ for randomly-established links.

[2]The DOP used in satellite navigation is usually defined in the form of $\sqrt{\mathrm{trace}[(G^{\mathrm{T}}G)^{-1}]}$ [24]. In this paper, we define DOP in the form of $\mathrm{trace}[(G^{\mathrm{T}}G)^{-1}]$ to simplify calculation and analysis.

A. Step 1: random node locations

Recall (6) which describes the elements in G. Note that when link k is incident to node n, i.e., $n \in \{i_k, j_k\}$,

$$\sum_{m=1}^{d}\Big(\frac{\partial r_k}{\partial p_{n,m}}\Big)^2 = \frac{\sum_{m=1}^{d}(p_{i_k,m} - p_{j_k,m})^2}{\|p_{i_k} - p_{j_k}\|^2} = 1. \tag{11}$$

Under the assumption that the nodes are randomly deployed such that $p_{i_k,m} - p_{j_k,m}$, $m = 1, \ldots, d$ are iid, we have

$$\mathrm{E}\Big(\frac{\partial r_k}{\partial p_{n,m}}\Big)^2 = \frac{1}{d}, \quad \forall m = 1, \ldots, d. \tag{12}$$

By (12), the elements of matrix $F = \{F_{\tilde{i}\tilde{j}}\} \in \mathbb{R}^{dN_S \times dN_S}$ have the conditional expectation

$$\begin{aligned}\Xi_{\tilde{i}\tilde{j}} &= \mathrm{E}_{\text{locations}}(F_{\tilde{i}\tilde{j}}|\text{connections}) = \mathrm{E}\sum_{k=1}^{K}\frac{\partial r_k}{\partial p_{i,m_1}}\frac{\partial r_k}{\partial p_{j,m_2}} \\ &= \begin{cases} \frac{1}{d}\deg(i) & \text{if } i = j \text{ and } m_1 = m_2, \\ -\frac{1}{d} & \text{if } (i,j) \in E \text{ and } m_1 = m_2, \\ 0 & \text{otherwise,} \end{cases}\end{aligned} \tag{13}$$

where $\tilde{i} = (i-1)d + m_1$, $\tilde{j} = (j-1)d + m_2$, $1 \le m_1, m_2 \le d$.

It can be seen that $\Xi = \breve{\Xi} \otimes I$, where $\otimes$ denotes the Kronecker product, and I is the identity matrix of size d. The elements of $\breve{\Xi}$ are given by

$$\breve{\Xi}_{ij} = \begin{cases} \frac{1}{d}\deg(i) & \text{if } i = j, \\ -\frac{1}{d} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The matrix $\breve{\Xi}$ describes how the network is connected, as its diagonal elements are given by $\breve{\Xi}_{ii} = \deg(i)/d$, and its non-zero off-diagonal element $\breve{\Xi}_{ij}$ indicates that there is a sensor-to-sensor link between nodes i and j. A surprising coincidence is that the matrix $d\breve{\Xi}$ is a submatrix of the graph Laplacian [27].

The lower bound of E-AGDOP (LB-E-AGDOP) can be calculated by inverting $F = \mathrm{E}\,\Xi$ or, equivalently, inverting $\breve{F} = \mathrm{E}\,\breve{\Xi}$, because $\mathrm{trace}[(\mathrm{E}\,\Xi)^{-1}] = d\,\mathrm{trace}[(\mathrm{E}\,\breve{\Xi})^{-1}]$.

B. Step 2: random link connections

Given an average degree δ, the trace of $\breve{\Xi}$ is given by

$$\mathrm{trace}(\breve{\Xi}) = \sum_{i=1}^{N_S}\breve{\Xi}_{ii} = \sum_{i=1}^{N_S}\deg(i)/d = N_S\delta/d. \tag{15}$$

Given an average sensor degree δ_S, there are $N_S\delta_S/2$ sensor-to-sensor links in the network, and thus $\breve{\Xi}$ includes $N_S\delta_S$ off-diagonal elements with a non-zero value of $-1/d$. Under the assumption that the sensor-to-sensor links are chosen uniformly at random from the set $\{(i,j)|1 \le i < j \le N_S, i, j \in \mathbb{Z}\}$, each off-diagonal element $\breve{\Xi}_{ij}$, $i \ne j$ satisfies the Bernoulli distribution

$$\breve{\Xi}_{ij} = \begin{cases} -1/d & \text{with probability } \frac{N_S\delta_S}{N_S(N_S-1)} = \frac{\delta_S}{N_S-1}, \\ 0 & \text{with probability } 1 - \frac{\delta_S}{N_S-1}. \end{cases} \tag{16}$$

Then, the expectation of $\check{F}$ is given by

$$\mathrm{E}\,\check{F}_{ij} = \mathrm{E}_{\text{links}}(\Xi_{ij}) = \begin{cases} \frac{\delta}{d} & \text{if } i = j, \\ -\frac{\delta_S}{d(N_S-1)} & \text{otherwise.} \end{cases} \tag{17}$$

Appendix II shows that

$$\mathrm{trace}[(\mathrm{E}\,\check{F})^{-1}] = \frac{N_S}{\eta}\Big(1 + \frac{\zeta}{1 - N_S\zeta}\Big), \tag{18}$$

where $\eta = d^{-1}[\delta + \delta_S/(N_S-1)]$ and $\zeta = \delta_S/[\delta(N_S-1)+\delta_S]$. Therefore, LB-E-AGDOP is given by

$$\begin{aligned} \text{LB-E-AGDOP} &= \frac{\mathrm{trace}[(\mathrm{E}\,F)^{-1}]}{N_S} = \frac{d}{\eta}\Big(1 + \frac{1}{\zeta^{-1} - N_S}\Big) \\ &= \frac{d^2}{\delta + \delta_S/(N_S-1)}\Big(1 + \frac{\delta_S}{\delta_A(N_S-1)}\Big) \\ &= \frac{d^2}{\delta}\frac{N_S - 1 + \delta_S/\delta_A}{N_S - 1 + \delta_S/\delta}. \end{aligned} \tag{19}$$

Thus far, we obtain a closed-form expression for LB-E-AGDOP. It depends on two parameters of network connectivity, δ_S and δ_A (note $\delta = \delta_S + \delta_A$), and one parameter of network scale, N_S. It can be seen that LB-E-AGDOP is approximately inversely proportional to the average degree, and a low average anchor degree worsens accuracy.

V. Accuracy for Large-Scale Networks

One of the advantages of sensor networks is that it can scale up by deploying more and more sensor nodes. As for range-based network localization, a question is whether the localization accuracy is still maintained when more sensor nodes join in the network, and what level of connectivity is required if we want to maintain the localization accuracy. In this section, we address these questions by analyzing how accuracy varies with the network scale for a certain level of connectivity.

As discussed in [16], [17], [23], $K_S \geq O(N_S)$ and $K_A \geq O(1)$ are necessary for the network to be localizable. Let us first consider the marginal case $K_S = O(N_S)$ and $K_A = O(1)$, which is equivalent to a constant δ_S and a constant K_A. Then, (19) can be written as

$$\begin{aligned} \text{LB-E-AGDOP} &= \frac{d^2}{K_A/N_S + \delta_S + \delta_S/(N_S-1)} \cdot \\ &\qquad \Big(1 + \frac{\delta_S}{(K_A/N_S)(N_S-1)}\Big) \\ &\approx \frac{d^2}{K_A/N_S + \delta_S}\Big(1 + \frac{\delta_S}{K_A}\Big) \\ &\to d^2\Big(\frac{1}{\delta_S} + \frac{1}{K_A}\Big) \qquad \text{as } N_S \to \infty, \end{aligned} \tag{20}$$

which indicates that increasing the number of sensor nodes deteriorates accuracy, with LB-E-AGDOP increasing monotonically towards a limit determined by δ_S and K_A.

Next, consider the case $K_S = O(N_S)$ and $K_A = O(N_S)$, which is equivalent to a constant δ_S and a constant δ_A. It can be calculated from (19) that as $N_S \to \infty$,

$$\text{LB-E-AGDOP} \to d^2/\delta. \tag{21}$$

TABLE I

Accuracy relationship to the connectivity and scale of the sensor network.

Network connectivity	As $N_S \nearrow \infty$
$K_S < O(N_S)$ or $K_A < O(1)$	no longer localizable
$K_S = O(N_S)$ and $K_A = O(1)$	AGDOP $\nearrow d^2(1/\delta_S + 1/K_A)$
$K_S = O(N_S)$ and $K_A = O(N_S)$	AGDOP $\searrow d^2/\delta$
$K_S = O(N_S^2)$ and $K_A = O(1)$	AGDOP $\searrow d^2/K_A$
$K_S = O(N_S^2)$ and $K_A > O(1)$	AGDOP $\searrow 0$

As the derivative of LB-E-AGDOP with respect to N_S

$$\frac{\partial \text{AGDOP}}{\partial N_S} = -\frac{d^2\delta_S^2}{\delta_A(\delta N_S - \delta_A)^2} < 0, \tag{22}$$

increasing the number of sensor nodes improves accuracy, with LB-E-AGDOP decreasing monotonically towards the limit determined by δ.

Furthermore, let us consider a very benign case that the sensor nodes form a complete graph [20], i.e., range measurements are available for every pair of distinct sensor nodes. Then, $\delta_S = N_S - 1$, $\delta = \delta_A + N_S - 1$, and (19) can be reduced to

$$\begin{aligned} \text{LB-E-AGDOP} &= \frac{d^2}{\delta_A + N_S}\Big(1 + \frac{1}{\delta_A}\Big) \\ &\to \begin{cases} d^2/K_A & \text{if } K_A = O(1) \\ 0 & \text{if } K_A > O(1) \end{cases} \end{aligned} \tag{23}$$

as $N_S \to \infty$. It can be seen that increasing the number of sensor nodes always improves accuracy. When $K_A = O(1)$, LB-E-AGDOP decreases monotonically towards d^2/K_A, just as if each sensor node is directly connected to all anchor nodes. When $K_A > O(1)$, LB-E-AGDOP decreases monotonically towards 0. Nevertheless, this good scalability is at a price of $O(N_S^2)$ sensor-to-sensor links.

Table I summaries the above findings. In general, range-based localization schemes can guarantee accuracy for large-scale network, even for the marginal case $K_S = O(N_S)$ and $K_A = O(1)$ which just ensures localizability.

VI. Simulation Results

In this section, we conduct numerical simulations to validate the theoretical results obtained in Sections IV and V. All simulation results presented in this section are based on the following settings.

- Two dimensions ($d = 2$);
- Sensor nodes are uniformly distributed in the unit square $[0,1] \times [0,1]$;
- Four anchors ($N_A = 4$) located at the corners of the unit square, i.e., $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$;
- Given K_S, sensor-to-sensor links are chosen uniformly at random from the set $\{(i,j) | 1 \leq i < j \leq N_S, i,j \in \mathbb{Z}\}$;
- Given K_A, anchor-to-sensor links are chosen uniformly at random from $V_S \times V_A$, where $V_S = \{1, 2, \ldots, N_S\}$ is the set of sensors and $V_A = \{N_S + 1, N_S + 2, \ldots, N\}$ is the set of anchors.

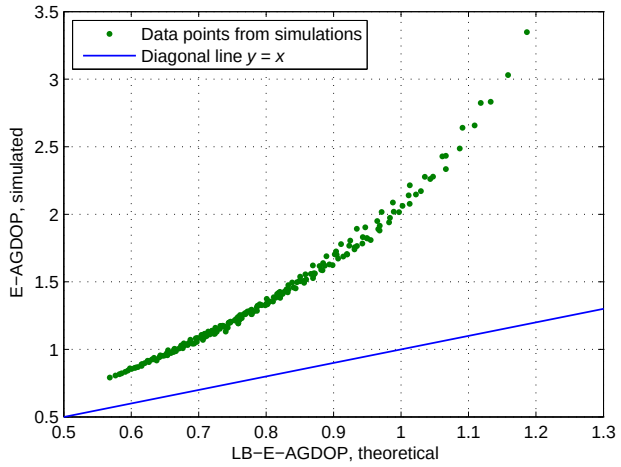

Fig. 2. Comparison between LB-E-AGDOP from (19) and E-AGDOP from simulations with the parameters $N_S = 10$, $K_S = 20, 21, \ldots, 35$, and $K_A = 8, 9, \ldots, 20$. The lower bound is tighter when E-AGDOP is smaller.

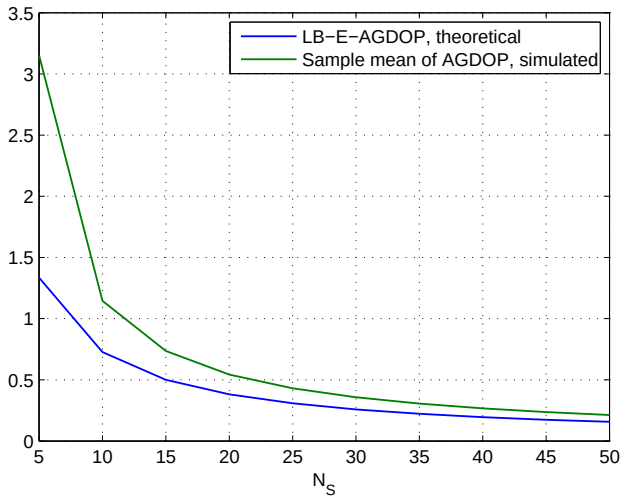

Fig. 3. Accuracy of range-based localization schemes for large-scale network. The sensor nodes form a complete graph ($K_S = N_S(N_S-1)/2$), and the number of anchor-to-sensor links is equal to the number of sensors ($K_A = N_S$).

Fig. 1 has shown a scenario excerpted from the simulation with the parameters $N_S = 10$, $K_S = 20$, and $K_A = 8$.

A. E-AGDOP and its theoretical lower bound

Fig. 2 compares LB-E-AGDOP from (19) and E-AGDOP obtained from simulations. The simulations are based on the parameters $N_S = 10$, $K_S = 20, 21, \ldots, 35$, and $K_A = 8, 9, \ldots, 20$. Each dot in Fig. 2 represents a network configuration with certain K_S and K_A.

It can be seen that our lower bound is tighter when E-AGDOP is smaller. Although the relationship between LB-E-AGDOP and E-AGDOP is nonlinear[3], if two different network configurations result in the same LB-E-AGDOP value, they also lead to very close E-AGDOP values. Therefore, our derived lower bound, LB-E-AGDOP, is a valid performance indicator for the accuracy of range-based localization schemes in random sensor networks. In particular, when LB-E-AGDOP approaches a certain limit, E-AGDOP also approaches a limit. This is an important foundation for our analysis in Section V.

B. Accuracy for large-scale network

As a validation of the theoretical results obtained in Section V, Fig. 3 shows how localization accuracy varies for an increasing network scale. The simulation is based on the benign case that the sensor nodes form a complete graph, and the number of anchor-to-sensor links is equal to the number of sensors. According to Table I, as $N_S \to \infty$, LB-E-AGDOP approaches 0, so does E-AGDOP. The simulation result confirms this conclusion, and also shows that the gap between LB-E-AGDOP and E-AGDOP decreases with an decreasing E-AGDOP.

VII. Concluding Remarks

This paper has studied the accuracy of range-based localization schemes in random sensor networks with respect to connectivity and the network scale. We have shown that the variance of localization errors are proportional to AGDOP. We have proved a novel lower bound of expectation of AGDOP and derived a closed-form formula (19) that relates LB-E-AGDOP to only three parameters: the average sensor degree, average anchor degree, and number of sensor nodes. We have then used this formula to study how localization accuracy varies with connectivity and the network scale. The following conclusions are drawn from our theoretical analysis.

- Localization accuracy is approximately inversely proportional to the average degree, and a low average anchor degree deteriorates accuracy.
- When network connectivity merely guarantees localizability, increasing sensor nodes leads to bounded monotonic increase in AGDOP. When a network is densely connected, increasing sensor nodes leads to bounded monotonic decrease in AGDOP.

The simulation results have validated the theoretical results, and shown that our derived lower bound, LB-E-AGDOP, is a valid performance indicator for the accuracy of range-based localization schemes in random sensor networks. The theory and results presented in this paper provide guidelines on designing range-based localization schemes for robotic sensor networks.

Appendix I
Proof of the Lower Bound of Expectation of DOP

The proof of the lower bound of expectation of DOP is based on the Cauchy–Schwarz inequality for the expectation of random matrices [29], [30].

Lemma 1 (Cauchy–Schwarz inequality for matrices): Let $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{n\times p}$ be random matrices such that $\mathrm{E}\,\|A\|^2 < \infty$, $\mathrm{E}\,\|B\|^2 < \infty$, and $\mathrm{E}(A^{\mathrm{T}}A)$ is non-singular. Then

$$\mathrm{E}(B^{\mathrm{T}}B) \succeq \mathrm{E}(B^{\mathrm{T}}A)[\mathrm{E}(A^{\mathrm{T}}A)]^{-1}\,\mathrm{E}(A^{\mathrm{T}}B), \qquad (24)$$

[3]In fact, E-AGDOP^{-1} is approximately a linear function of LB-E-AGDOP^{-1} [28].

where the operator $X \succeq Y$ means that $X - Y$ is positive semidefinite.

With the substitutions $A = G$ and $B = G(G^{\mathrm{T}}G)^{-1}$ into the above inequality, we have the following theorem.

Theorem 1: For a random network with a non-singular geometry matrix G defined in (5),

$$U = \mathrm{E}[(G^{\mathrm{T}}G)^{-1}] \succeq V = [\mathrm{E}(G^{\mathrm{T}}G)]^{-1}. \tag{25}$$

Since the diagonal elements of a positive semidefinite matrix must be non-negative, we have

$$U_{ii} \geq V_{ii}, \quad \forall i = 1, \ldots, dN_S, \tag{26}$$

where $U = [U_{ij}]$ and $V = [V_{ij}]$. In particular, the expectation of GDOP, $\mathrm{trace}(U)$, has a lower bound $\mathrm{trace}(V)$.

Appendix II
Proof of Eq. (18)

Lemma 2 (Sherman–Morrison formula [31]): Suppose A is an invertible square matrix, and u and v are vectors. Suppose furthermore that $1 + v^{\mathrm{T}}A^{-1}u \neq 0$. Then the Sherman–Morrison formula states that

$$(A + uv^{\mathrm{T}})^{-1} = A^{-1} - \frac{A^{-1}uv^{\mathrm{T}}A^{-1}}{1 + v^{\mathrm{T}}A^{-1}u}. \tag{27}$$

With $\eta = d^{-1}[\delta + \delta_S/(N_S - 1)]$, (17) can be written as

$$\eta^{-1}\,\mathrm{E}\,\check{F} = I - uu^{\mathrm{T}}, \tag{28}$$

where $u = \sqrt{\zeta}(1, 1, \ldots, 1)^{\mathrm{T}}$, and $\zeta = \delta_S/[\delta(N_S - 1) + \delta_S]$.

Letting $u = -v = \sqrt{\zeta}(1, 1, \ldots, 1)^{\mathrm{T}}$, by the Sherman–Morrison formula we have

$$(I - uu^{\mathrm{T}})^{-1} = I + uu^{\mathrm{T}}/(1 - u^{\mathrm{T}}u), \tag{29}$$

and thus

$$\begin{aligned} \eta\,\mathrm{trace}\big[(\mathrm{E}\,\check{F})^{-1}\big] &= \mathrm{trace}[(I - uu^{\mathrm{T}})^{-1}] \\ &= N_S + N_S\zeta/(1 - N_S\zeta). \end{aligned} \tag{30}$$

References

[1] I. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, "Wireless sensor networks: a survey," *Computer Networks*, vol. 38, no. 4, pp. 393–422, 2002.

[2] K. Langendoen and N. Reijers, "Distributed localization in wireless sensor networks: a quantitative comparison," *Comput. Netw.*, vol. 43, no. 4, pp. 499–518, Nov. 2003.

[3] J. Cortés, S. Martínez, T. Karataş, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.

[4] J. Heidemann, W. Ye, J. Wills, A. Syed, and Y. Li, "Research challenges and applications for underwater sensor networking," in *Proceedings of the 2006 IEEE Wireless Communications and Networking Conference (WCNC 2006)*, vol. 1, Apr. 2006, pp. 228–235.

[5] D. Moore, J. Leonard, D. Rus, and S. Teller, "Robust distributed network localization with noisy range measurements," in *Proceedings of the 2nd international conference on Embedded networked sensor systems (SenSys '04)*. New York, NY, USA: ACM, 2004, pp. 50–61.

[6] C. Wu, Y. Zhang, W. Sheng, and S. Kanchi, "Rigidity guided localisation for mobile robotic sensor networks," *Int. J. Ad Hoc Ubiquitous Comput.*, vol. 6, no. 2, pp. 114–128, Jul. 2010.

[7] R. Peng and M. L. Sichitiu, "Angle of arrival localization for wireless sensor networks," in *Proceedings of the 3rd Annual IEEE Communications Society Conference on Sensor, Mesh and Ad Hoc Communications and Networks (SECON 2006)*, vol. 1, Sep. 2006, pp. 374–382.

[8] J. Bruck, J. Gao, and A. A. Jiang, "Localization and routing in sensor networks by local angle information," *ACM Trans. Sen. Netw.*, vol. 5, no. 1, pp. 7:1–7:31, Feb. 2009.

[9] Y. Shang, W. Ruml, Y. Zhang, and M. P. J. Fromherz, "Localization from mere connectivity," in *Proceedings of the 4th ACM international symposium on Mobile ad hoc networking & computing (MobiHoc '03)*. New York, NY, USA: ACM, 2003, pp. 201–212.

[10] Y. Wang, X. Wang, D. Wang, and D. Agrawal, "Range-free localization using expected hop progress in wireless sensor networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 20, no. 10, pp. 1540–1552, Oct. 2009.

[11] K. Römer, "The lighthouse location system for smart dust," in *Proceedings of the 1st international conference on Mobile systems, applications and services (MobiSys '03)*. New York, NY, USA: ACM, 2003, pp. 15–30.

[12] Z. Zhong and T. He, "Sensor node localization with uncontrolled events," *ACM Trans. Embed. Comput. Syst.*, vol. 11, no. 3, pp. 65:1–65:25, Sep. 2012.

[13] M. W. M. G. Dissanayake, P. Newman, S. Clark, H. Durrant-Whyte, and M. Csorba, "A solution to the simultaneous localization and map building (SLAM) problem," *IEEE Transactions on Robotics and Automation*, vol. 17, no. 3, pp. 229–241, 2001.

[14] C. Taylor, A. Rahimi, J. Bachrach, H. Shrobe, and A. Grue, "Simultaneous localization, calibration, and tracking in an ad hoc sensor network," in *Proceedings of the 5th international conference on Information processing in sensor networks*, ser. IPSN '06. New York, NY, USA: ACM, 2006, pp. 27–33. [Online]. Available: http://doi.acm.org/10.1145/1127777.1127785

[15] V. Chandrasekhar, W. K. Seah, Y. S. Choo, and H. V. Ee, "Localization in underwater sensor networks: survey and challenges," in *Proceedings of the 1st ACM international workshop on Underwater networks (WUWNet '06)*. New York, NY, USA: ACM, 2006, pp. 33–40.

[16] B. Jackson and T. Jordán, "Connected rigidity matroids and unique realizations of graphs," *Journal of Combinatorial Theory, Series B*, vol. 94, no. 1, pp. 1–29, 2005.

[17] J. Aspnes, T. Eren, D. K. Goldenberg, A. S. Morse, W. Whiteley, Y. R. Yang, B. D. O. Anderson, and P. N. Belhumeur, "A theory of network localization," *IEEE Transactions on Mobile Computing*, vol. 5, no. 12, pp. 1663–1678, Dec. 2006.

[18] Y. Ding, N. Krislock, J. Qian, and H. Wolkowicz, "Sensor network localization, Euclidean distance matrix completions, and graph realization," *Optimization and Engineering*, vol. 11, pp. 45–66, 2010.

[19] D. B. West, *Introduction to Graph Theory*, 2nd ed. Prentice Hall, 2001.

[20] R. Diestel, *Graph Theory*, 4th ed. Springer-Verlag, Heidelberg, Jul. 2010.

[21] M. F. i Azam and M. N. Ayyaz, *Wireless Sensor Networks: Current Status and Future Trends*. CRC Press, Nov. 2012, ch. Location and Position Estimation in Wireless Sensor Networks, pp. 179–214.

[22] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilisation of infinitesimally rigid formations of multi-robot networks," *International Journal of Control*, vol. 82, no. 3, pp. 423–439, 2009.

[23] L. Heng and G. X. Gao, "Poster abstract: Range-based localization in sensor networks: localizability and accuracy," in *Proceedings of the International Conference on Information Processing in Sensor Networks (IPSN '13)*, Philadelphia, PA, 2013, pp. 329–330.

[24] P. Misra and P. Enge, *Global Positioning System: Signals, Measurements, and Performance*, 2nd ed. Lincoln, MA: Ganga-Jamuna Press, 2006.

[25] E. G. Larsson, "Cramér-Rao bound analysis of distributed positioning in sensor networks," *IEEE Signal Processing Letters*, vol. 11, no. 3, pp. 334–337, Mar. 2004.

[26] F. Penna, M. A. Caceres, and H. Wymeersch, "Cramér-Rao bound for hybrid gnss-terrestrial cooperative positioning," *IEEE Communications Letters*, vol. 14, no. 11, pp. 1005–1007, Nov. 2010.

[27] W. N. Anderson and T. D. Morley, "Eigenvalues of the Laplacian of a graph," *Linear and Multilinear Algebra*, vol. 18, no. 2, pp. 141–145, 1985.

[28] L. Heng and G. X. Gao, "Accuracy of range-based localization in random sensor networks," *IEEE Transactions on Signal Processing*, Jun. 2013, submitted.

[29] G. Tripathi, "A matrix extension of the Cauchy-Schwarz inequality," *Economics Letters*, vol. 63, no. 1, pp. 1–3, 1999.

[30] P. Lavergne, "A Cauchy-Schwarz inequality for expectation of matrices," Simon Fraser University, Tech. Rep., Nov. 2008.

[31] W. W. Hager, "Updating the inverse of a matrix," *SIAM Review*, vol. 31, no. 2, pp. pp. 221–239, Jun. 1989.